Remarks on Lattice Gauge Fixing

S. Petrarca
Dipartimento di Fisica, Università di Roma "La Sapienza",
P.le A. Moro 2, I-00185 Roma, Italy.
INFN, Sezione di Roma 1, P.le A. Moro 2, I-00185 Roma, Italy.
P.le A. Moro 2, I-00185 Roma, Italy.

In this talk I briefly comment on the conventional lattice gauge fixing adopting a critical, even though constructive, numerical point of view.

1 Standard Landau

On the lattice the usual procedure accepted to compute gauge dependent matrix elements is summarized in the following formula defining the expectation value of a gauge dependent operator $\mathcal{O}$:

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int dU \mathcal{O}(U^G) e^{-\beta S(U)} , \tag{1}$$

where $S(U)$ is the Wilson lattice gauge invariant action, G is the gauge transformation projecting the links in the Landau gauge

$$\partial_\mu A_\mu^G = 0 \; + \; \text{periodic boundary condition} \tag{2}$$

with the gauge rotation given by $U_\mu^G(x) = G(x) U_\mu(x) G^\dagger(x+\mu)$ and the gluon field defined on the lattice in the standard way:

$$A_\mu(x) \equiv \left[\frac{U_\mu(x) - U_\mu^\dagger(x)}{2iag_0} \right]_{Traceless} . \tag{3}$$

In the lattice gauge theories where the links belong to a compact group, the gauge fixing is necessary only in the case of the measure of a gauge dependent operator. An expression similar to (1) but without having gauge fixed the links, defines the expectation value of a gauge independent operator. Moreover, in a lattice simulation there is no need to compute the Faddeev-Popov determinant because the correct adjustment of the measure, necessary in the case of gauge fixing, is obtained rotating the links in the chosen gauge. Therefore the complexity of the ghost technique is replaced by the numerical evaluation of the gauge transformations. The price to pay is the large amount of computer time spent to obtain numerically the gauge transformations. From a numerical point of a view

the values of a gauge dependent operators strongly fluctuate around zero if the gauge has not been fixed. In the case of an imperfect or inadequate gauge fixing the measure of a gauge dependent operator is affected by additional fluctuations to be summed up to the intrinsic statistical noise.

The necessary steps bringing to the computation of the integral (1) can be described as follows:

- A set of N thermalized configurations $\{U\}$ is generated with periodic boundary conditions according to the gauge invariant weight $e^{-S_W(U)}$;
- For each $\{U\}$ a numerical algorithm compute the gauge transformation G;
- The expectation value of an operator is given by the mean value of the values taken by the operator on the gauge rotated configurations:

$$\langle \mathcal{O} \rangle^{Latt} = \frac{1}{N} \sum_{\{conf\}} \mathcal{O}(U^G) \quad . \tag{4}$$

The gauge fixing algorithm is based on the minimization of a functional $F_U[G]$ constructed in such a way that its extrema are the gauge fixing transformations corresponding to the gauge condition. The F standard form for the Landau gauge is:

$$F_U[G] = -Re\ Tr\ \sum_{\mu,x} {U_\mu}^{G(x)}(x) \tag{5}$$

and the transformations G for which $\frac{\delta F}{\delta G} = 0$ rotate the links in the gauge $\partial_\mu A_\mu^G = 0$. The algorithm sweeps all the lattice many times and it stops when a prefixed quality factor is reached.

It is remarkable that the eq. (5) does not correspond to the natural discretization of the continuum functional

$$F_A[G] \equiv -\ Tr\ \int d^4x\ \left(A_\mu^G(x) A_\mu^G(x)\right) \equiv -\left(A^G, A^G\right) \equiv -||A^G||^2\ , \tag{6}$$

according to the lattice definition of the gluon field (3) but it differs from that by $O(a)$ terms. The form in eq. (5) is adopted not only for its simplicity but also because its minimization enforces the following discretized version of the gauge condition

$$\Delta^G(x) \equiv \sum_{\mu=1}^{4} \left(A_\mu^G(x) - A_\mu^G(x-\hat{\mu})\right) = 0 \tag{7}$$

where A_μ must be related to the links by the standard definition (3).

In order to study the approach to the minimum, two quantities are usually monitored. The first one is $F[U^G]$ itself, which decreases monotonically and eventually reaches a plateau. The other one, denoted by θ, is defined as follows:

$$\theta^G \equiv \frac{1}{V}\ \sum_n \theta^G(x) \equiv \frac{1}{V}\ \sum_n Tr\ [\Delta^G(x)(\Delta^G)^\dagger(x)], \simeq \int d^4x\ Tr(\partial_\mu A_\mu^G)^2, \tag{8}$$

where V is the lattice volume.

The function θ decreases (not strictly monotonically) approaching zero when $F_U[G]$ reaches its minimum. The desired gauge fixing quality is determined stopping the computer code when θ^G has achieved a preassigned value close to zero. The choice of the gauge fixing quality is a delicate point in the case of a simulation with a large volume and a high number of thermalized configurations. Of course, the better is the gauge fixing quality, the more computer time is needed. Moreover it is impossible to know before computing the gauge dependent correlation functions if the choice done is suitable. So that, the stopping θ value is normally fixed on the basis of a practical compromise between the estimated computer time and the gauge fixing quality. Sometimes, in the case of calculations performed on computers with single precision floating point, the maximum gauge fixing quality is limited by a value of the order of the floating point zero: $\theta \simeq 10^{-7}$, this value is usually enough to guarantee the stability of gauge dependent correlators.

2 Lattice Gribov Copies

On the lattice a conceptual and numerical difficulty connected with gauge fixing is the existence of many different minima of the functional $F[U]$. The different gauge transformations determined by different minima are not equivalent each other and can be labelled with the value of the functional F. Of course it is unthinkable to succeed in reaching numerically the absolute minimum. The search of the F minima is at least as difficult as to find the lowest state of energy of a spin glass system with hamiltonian F. So that the condition (2) does not fix the gauge in a complete way generating on the lattice a problem analogous to the Gribov copies in the continuum [1, 2]. However the analogy is only formal because it is not possible to establish a connection among continuum and lattice copies. Moreover it is also likely that (many) lattice Gribov copies are spurious solutions due to the discretization [3].

Actually, it must be noted that the presence of Gribov copies in the lattice Landau gauge fixing is a very common phenomenon. For example, it has been shown that when the minimization algorithm includes the over-relaxation technique, varying the value of the over-relaxation parameter ω different lattice Gribov copies [4] are generated. Of course there is no correlation between the convergence rate and the value of F associated with the particular Gribov copy found.

The numerical effects of lattice Gribov copies can be divided into two categories: the distortion of a measurement and the lattice Gribov noise. The typical example of a distortion due to the existence of Gribov copies is the measure of the photon propagator in compact $U(1)$ in the so called Coulomb phase. In this case the measure of the photon propagator as function of the momentum, performed using the gauge fixing in the standard way, was affected by a not

regular behaviour [5]. This problem was associated with the distortional effects due to the Gribov copies. In fact, after having chosen the gauge fixed configurations nearest to the minimum of the gauge functional, the photon propagator became a smooth momentum function. More recent studies [6] show the details of the Gribov copies dynamics and provide a practical procedure to eliminate their effects. It is interesting to note that in the case of the measure of the gluon propagator in $SU(3)$ there is no signal in the literature about a similar problem (for a recent review see ref. [7]). The numerical simulations are performed in the Landau gauge and the various authors claim that the effects of Gribov copies do not affect the measure.

Anyway, in the normal case in which there is no distortion due to Gribov copies, there should be an increase of the numerical fluctuations due to the incomplete gauge fixing associated with the copies. An attempt to study the properties of this noise has been done in ref. [8] taking as an example the measurement of the lattice axial current Z_A. This quantity is particularly well suited to the study of the Gribov fluctuations, because it is a gauge independent quantity but it can be obtained from chiral Ward identities in two distinct ways: a gauge independent one, which consists in taking the matrix elements between hadronic states without fixing the gauge in the simulation, and a gauge dependent one, which consists in taking the matrix elements between quark states in the Landau gauge. In the intermediate steps of the numerical computation, the second procedure takes into account gauge dependent matrix elements potentially subjected to the Gribov noise. Hence, there is an explicitly gauge invariant estimate of Z_A which is free of Gribov noise and which can be directly compared to the gauge dependent, Gribov affected, estimate.

The results of the analysis can be summarized in the following way:

- there is a clear evidence of residual gauge freedom associated with lattice Gribov copies;
- the lattice Gribov noise is not separable from the statistical uncertainty of the Monte Carlo method.

The global effect is not dramatic because the Z_A value obtained with the gauge dependent methods (1.08(5)) is close to the gauge independent evaluation (1.06(6)) and the jacknife errors are comparable.

3 Gluon Field Definition

In order to impose the Landau gauge on the lattice it is necessary to define the gluon field A_μ in terms of the links. It is clear that the definition given in eq. (3) is far from unique and it cannot be preferred, from the first principles, to any other definition with analogous properties. Moreover, in the general field theoretical

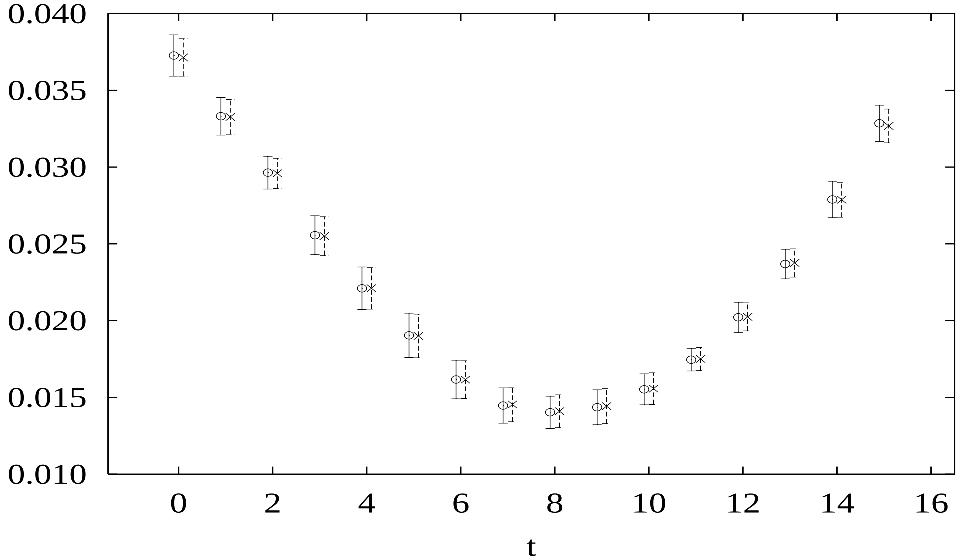

Figure 1: Comparison of the matrix elements of $\langle \mathcal{A}_i^{'} \mathcal{A}_i^{'} \rangle(t)$ (crosses) and the rescaled $\langle \mathcal{A}_i \mathcal{A}_i \rangle \cdot C_i^2(g_0)$ (open circles) as function of time for a set of 50 thermalized $SU(3)$ configurations at $\beta = 6.0$ with a volume $V \cdot T = 8^3 \cdot 16$. The data have been slightly displaced in t for clarity, the errors are jacknife.

framework any pair of operators differing from each other by irrelevant terms, i.e. formally equal up to terms of order a, will tend, to the same continuum operator, up to a constant. It is has been shown in ref. [9] that this feature is satisfied at the non-perturbative level in lattice QCD. In fact, different definitions of the gluon field, at the lattice level, give rise to Green's functions proportional to each other, thus guaranteeing the uniqueness of the renormalized continuum gluon field. The relation between the two A_μ definitions can be expressed up to $O(a^2)$ terms in this way [10]:

$$A^{'}{}_\mu(x) = C(g_0) A_\mu(x). \tag{9}$$

Therefore for a Green's functions insertions the following ratio is expected to be a constant

$$\frac{\langle \ldots A_\mu^{\prime}(x) \ldots \rangle}{\langle \ldots A_\mu(x) \ldots \rangle} = C(g_0) \ . \tag{10}$$

This relation has been checked numerically on the lattice by measuring a set of Green functions related to the gluon propagator for $SU(3)$ in the Landau gauge with periodic boundary conditions. In Fig. 1 the Green functions $\langle \mathcal{A}_i^{\prime} \mathcal{A}_i^{\prime} \rangle$ and the rescaled one $C_i^2(g_0) \langle \mathcal{A}_i \mathcal{A}_i \rangle$ are shown, where

$$\langle \mathcal{A}_i \mathcal{A}_i \rangle(t) \equiv \frac{1}{3V^2} \sum_i \sum_{\mathbf{x},\mathbf{y}} Tr \langle A_i(\mathbf{x}, t) A_i(\mathbf{y}, 0) \rangle \tag{11}$$

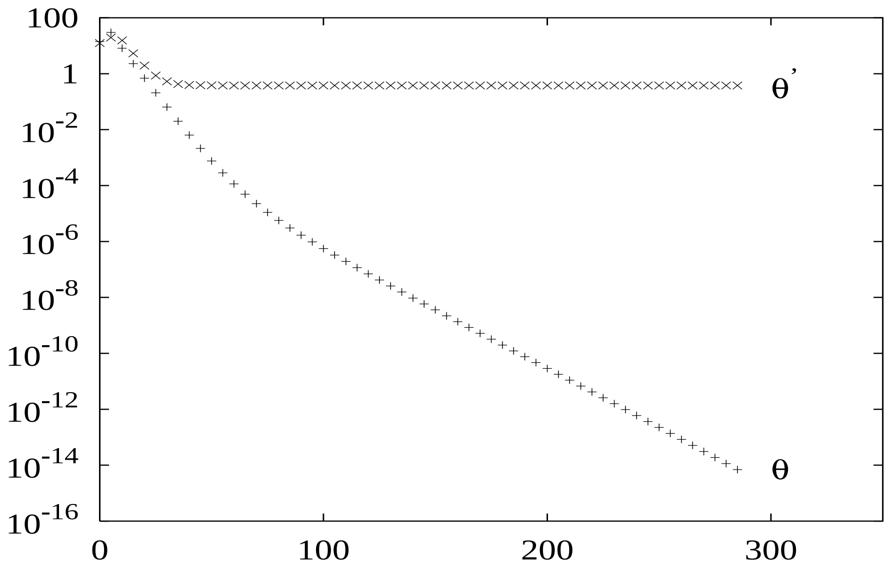

Figure 2: Typical behaviour of θ and $\theta^{'}$ vs gauge fixing sweeps at $\beta = 6.0$ for a thermalized $SU(3)$ configuration $8^3 \cdot 16$.

and the operator $\langle \mathcal{A}_i^{'} \mathcal{A}_i^{'} \rangle (t)$ is obtained replacing in the same form the alternative definition:

$$A^{'}{}_{\mu}(x) \;\equiv\; \frac{((U_\mu(x))^2 - (U_\mu^\dagger(x))^2)_{traceless}}{4iag_0}, \quad \mu = 1, \ldots 4. \tag{12}$$

The remarkable agreement between these two quantities confirms the proportionality shown in eq. (9).

From the numerical point of view, however, the various definitions are not interchangeable. In fact let me suppose to fix the gauge of a thermalized configuration stopping the gauge-fixing sweeps when $\theta \leq 10^{-14}$ and then define $\theta^{'}$ as having the same functional form of θ, as in eq. (8), but with A_μ replaced by $A_\mu^{'}$ given in eq. (12) The values of θ and $\theta^{'}$ during the minimization of F are reported in Fig. 2, for a typical thermalized configuration, as functions of the lattice sweeps of the numerical gauge-fixing algorithm. As clearly seen $\theta^{'}$ does not follow the same decreasing behaviour as θ: after an initial decrease, $\theta^{'}$ goes to a constant value, many orders of magnitude higher than the corresponding value of θ. This difference, already noted in ref. [3], between the behavior of θ and $\theta^{'}$ could cast some doubts on the lattice gauge-fixing procedure and on the corresponding continuum limit of gauge dependent operators.

This paradoxical situation is due to the fact that $\theta^{'}$ is an operator and the lattice can attribute a value to it only after averaging it over the gauge fixed configurations of the thermalized set. Hence the comparison reported in Fig. 2 is

devoid of meaning because it is done comparing the values obtained by a single configuration. Moreover the behavior shown in Fig. 2 can be readily understood in the following way. The operator θ, defined in eq. (8) ($\theta^{'}$), is computed in the lattice units taking the definition eq.(3) (eq.(12)) without the powers of a to the denominator. Then in the continuum variables $\theta = \frac{a^4}{\mathcal{V}} \int d^4x (\partial_\mu A_\mu(x))^2$ where $\mathcal{V}$ is the 4-volume in physical units (analogously for $\theta^{'}$). Hence, while θ vanishes configuration by configuration, as a consequence of the gauge fixing, $\theta^{'}$ is proportional to $(\partial_\mu A^{\prime}_\mu)^2$, which has the vacuum quantum numbers and mixes with the identity. The expectation value of $(\partial_\mu A^{\prime}_\mu)^2$, therefore, diverges as $\frac{1}{a^4}$ so that $\theta^{'}$ will stay finite, as $a \to 0$.

4 Summary and Addendum

Every step of the usual gauge fixing procedure is affected by subtleties. The Gribov copies can be moderately dangerous in a simulation but it is necessary to check their influence in any calculation. The definition of the gluon field in terms of the links is not a fixed prescription of the theory but it can be chosen, for example, in order to satisfy practical requests.

It is also possible to take advantage from this freedom as it has been done in ref. [11, 12] in order to implement a procedure to fix a generic covariant gauge on the lattice. The great advantage of a covariant gauge is that varying the value of the gauge parameter it is possible to check numerically, in the calculation of gauge dependent Green's functions like for example the gluon propagator, the gauge dependence of the fitted parameters.

After the end of this workshop a thorough study of the lattice covariant gauges and their applications has been completed [13].

5 Acknowledgements

I thank Valya Mitrjushkin and all the organizers of the workshop "Lattice Fermions and Structure of the Vacuum" for the warm hospitality and the excellent organization.

References

[1] Gribov, V. N. (1978) *Nucl. Phys.* **B139**, 1.

For a review of the Gribov ambiguity in the continuum see for example Sciuto, S. (1979) *Phys. Rep.* **49**, 181, and references therein.

[2] van Baal, P. (1997) in NATO Avanced Study Institute on Confinement, Duality and Nonperturbative Aspects of QCD, Cambridge, England, 23 June - 4 July, 1997, hep-th/9711070.

[3] Giusti, L. (1997) *Nucl. Phys.* **B498**, 331.

[4] Paciello, M. L., Parrinello, C., Petrarca, S., Taglienti, B., Vladikas, A. (1992) *Phys. Lett.* **B276**, 163.

[5] Nakamura, A. and Plewnia, M. (1991) *Phys. Lett.* **B255**, 274.

[6] Bogolubsky, I. L., Mitrjushkin, V. K., Muller-Preussker, M., Peter, P. (1999) *Phys. Lett.* **B458**, 102; Lorentz gauge fixing and the Gribov problem: the fermion correlator in lattice compact QED with Wilson fermions, hep-lat/9910037 and references therein.

[7] Mandula, J. E. (1999) The gluon propagator, hep-lat/9907020.

[8] Paciello, M. L., Parrinello, C., Petrarca, S., Taglienti, B., Vladikas, A. (1994) *Phys.Lett.* **B341**, 187.

[9] Giusti, L., Paciello, M. L., Petrarca, S., Taglienti, B., Testa, M. (1998) *Phys. Lett.* **B432**, 196.

[10] Testa, M. (1998) *JHEP* 9804:002.

[11] Giusti, L., Paciello, M. L., Petrarca, S., Taglienti, B. (1999) How to fix nonperturbatively a parameter dependent covariant gauge on the lattice, presented at Lattice99, Pisa, Italy, June 29 - July 3, 1999, hep-lat/9910012.

[12] Giusti, L., Paciello, M. L., Petrarca, S., Taglienti, B. (1999) Preliminary results with lattice covariant gauge, poster presented at this workshop: NATO Avanced Research Workshop on Lattice Fermions and Structure of the Vacuum, Dubna, Russia, October 5-9, 1999.

[13] Giusti, L., Paciello, M. L., Petrarca, S., Taglienti, B. (1999) Lattice gauge fixing for parameter dependent covariant gauges, submitted to *PRD*, hep-lat/9911038.